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CERTAIN PROBLEMS OF AN ARBITRARILY ORIENTED STRINGER IN A COMPOSITE ANISOTROPIC PLANE*

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The fundamental solution for a composite anisotropic plane is constructed taking defects into account on the line separating the materials. On this basis, a mathematical formulation is given of the contact problem for an arbitrarily oriented stringer located in one of the half-planes, in the form of a singular integral equation with a fixed singularity. The solvability of the equation obtained is proved. The power nature of the behaviour of the solution is clarified (by using the asymptotic properties of Cauchy-type integrals), and the conditions are established for the satisfaction of which the mentioned asymptotic form is strengthened by a logarithmic polynomial. An exact solution is constructed for the problem of a semi-infinite stringer leaving the line of material separation at an arbitrary angle, and an analogous problem for a finite inextensible stringer. It is hown that the asymptotic form of the contact shear stresses at the point of departure does not depend on the elastic properties of the stringer.

Solutions of analogous problems for anisotropic and orthotropic half-planes can be obtained by a passage to the limit. Such problems were examined earlier for a finite stringer in an anisotropic half-plane /1/ and for a semi-infinite stringer perpendicular to the boundary of an orthotropic half-plane /2, 3/. An incorrect assumption was made here about the power-logarithmic nature of the behaviour of the solution at the point of stringer departure at the boundary (the authors incorrectly utilized the asymptotic properties of Cauchy-type integrals). The error of the result /2/ is mentioned in /3/ where an exact solution of the problem is constructed. However, the true reason for the error is not given here, in which connection a false deduction is made about the fact that the asymptotic form of Cauchy-type integrals does not permit a unique solution of the question of the nature of the singularity if it is assumed to be power-logarithmic. The error of this deduction is shown below.

1. Construction of the fundamental solution.

Consider a piecewise-homogeneous plane consisting of two different anisotropic half-planes $(x \gtrless 0, |y| < \infty)$ not completely contacting along the line $x = 0$. Four of the following quantities

$$
\begin{aligned}
&H_n^{\pm}(y) = \varphi_n^{+}(y) \pm \varphi_n^{-}(y) \quad (n = 2, 3, 4, 5; y \in L) \\
&\varphi_2^{\pm}(y) = \sigma_x(\pm 0, y), \quad \varphi_3^{\pm}(y) = \tau_{xy}(\pm 0, y), \\
&\varphi_4^{\pm}(y) = \partial_2^1 v(\pm 0, y) \\
&\varphi_5^{\pm}(y) = \partial_2^1 u(\pm 0, y) \quad \left(\partial_k^m \equiv \frac{\partial^m}{\partial x_k^m}, k = 1, 2; x_1 \to x, x_2 \to y\right)
\end{aligned}
\tag{1.1}
$$

are considered to be known on sections where there is no direct interaction between the half-planes $(x = 0, y \in L)$.

Let a concentrated force with the projections P_1 and P_2 on the x and y coordinate axes, respectively, be applied to an arbitrary point (x_0, y_0) of the plane. It is required to find the components of the stress and strain state of the plane.

We will call the matrix $U = \{u_{nj}(x, y, x_0, y_0)\}$ $(n = 1, 2, \ldots, 5; j = 1, 2)$ the fundamental solution (in the sense of the theory of generalized functions) of the plane problem for a composite anisotropic medium if its components satisfy the system of equations

$$
\begin{aligned}
&\partial_1^1 u_{2j} + \partial_2^1 u_{3j} = \delta_{1j}\delta(x - x_0, y - y_0) \\
&\partial_1^1 u_{3j} + \partial_2^1 u_{1j} = \delta_{2j}\delta(x - x_0, y - y_0)
\end{aligned}
\tag{1.2}
$$

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$$\partial_1^{1}u_{5j}=a_{11}u_{2j}+a_{12}u_{1j}+a_{16}u_{3j}$$
$$\partial_2^{1}u_{4j}=a_{12}u_{2j}+a_{22}u_{1j}+a_{26}u_{3j}$$
$$\partial_2^{1}u_{5j}+\partial_1^{1}u_{4j}=a_{16}u_{2j}+a_{26}u_{1j}+a_{66}u_{3j}$$

$a_{mn}=\theta(x)a_{mn}^{+}+\theta(-x)a_{mn}^{-}$, δ_{mn} is the Kronecker delta, $\delta(x, y)$ and $\theta(x)$ are the Dirac and Heaviside functions, and a_{mn}^{+}, a_{mn}^{-} are coefficients of the generalized Hooke's law /4/ for the half-planes $x>0$ and $x<0$, respectively.

For $j=1$ system (1.2) describes the stress and strain state of a plane due the action of a single concentrated force in the direction of the x-axis, and for $j=2$ in the direction of the y-axis.

By virtue of (1.1) the components of the matrix U satisfy the following conditions on the line of connection of the materials $x=0$

$$u_{nj}(+0,y,x_0,y_0)\pm u_{nj}(-0,y,x_0,y_0)=H_{nj}^{\pm}(y) \qquad (1.3)$$
$$(n=2,3,4,5;\ y\in L)$$

We apply a Fourier integral transform in y (with the parameter β) and a generalized transform /5/ in x (with parameter α) to the system (1.2). Introducing the notation

$$v_{nj}^{+}=\int_0^\infty u_{nj}^{*}(x)e^{i\alpha x}dx,\quad v_{nj}^{-}=-\int_{-\infty}^{0}u_{nj}^{*}(x)e^{i\alpha x}dx,$$
$$u_{nj}^{*}(x)=\int_{-\infty}^{\infty}u_{nj}(x,y)e^{i\beta y}dy$$

for the semitransform, and relying on the known properties /6/ of the latter, we obtain a Riemann matrix problem in the semitransform of the stress tensor components

$$B^{+}(\alpha)V^{+}(\alpha)=B^{-}(\alpha)V^{-}(\alpha)+F(\alpha)\quad(-\infty<\alpha<\infty) \qquad (1.4)$$
$$V^{\pm}(\alpha)=\begin{Vmatrix}v_{2j}^{\pm}\\ v_{1j}^{\pm}\\ v_{3j}^{\pm}\end{Vmatrix},\quad B^{\pm}(\alpha)=\begin{Vmatrix}-i\alpha & 0 & -i\beta\\ 0 & -i\beta & -i\alpha\\ l_1^{\pm} & l_2^{\pm} & l_3^{\pm}\end{Vmatrix}$$
$$l_1^{\pm}(\alpha,\beta)=a_{12}^{\pm}\alpha^2-a_{16}^{\pm}\alpha\beta+a_{11}^{\pm}\beta^2,\quad l_2^{\pm}(\alpha,\beta)=a_{22}^{\pm}\alpha^2-a_{26}^{\pm}\alpha\beta+a_{16}^{\pm}\beta^2$$
$$F(\alpha)=\begin{Vmatrix}-\delta_{1j}\exp[i(\alpha x_0+\beta y_0)]+h_{2j}^{-}\\ -\delta_{2j}\exp[i(\alpha x_0+\beta y_0)]+h_{3j}^{-}\\ (-i\beta)^2h_{5j}^{-}+\alpha\beta h_{4j}^{-}\end{Vmatrix}$$
$$l_3^{\pm}(\alpha,\beta)=a_{26}^{\pm}\alpha^2-a_{66}^{\pm}\alpha\beta+a_{16}^{\pm}\beta^2,\quad h_{nj}^{\pm}(\beta)=\int_{-\infty}^{\infty}H_{nj}^{\pm}(y)e^{i\beta y}dy$$

The solution (1.4) is constructed successfully by reducing the problem to the solution of a Riemann scalar problem. To this end, we first consider the following problem about a jump:

$$v_{nj}^{+}(\alpha)-v_{nj}^{-}(\alpha)=F_{nj}(\alpha)\quad(-\infty<\alpha<\infty,\ n=1,2)$$
$$F_{nj}(\alpha)=-i\frac{\alpha^{1-n}}{\beta^{2-n}}\delta_{3-n,j}\exp[i(\alpha x_0+\beta y_0)]-\left(\frac{\alpha}{\beta}\right)^{3-2n}v_{3j}(\alpha)+i\frac{\alpha^{1-n}}{\beta^{2-n}}h_{4-n,j}^{-}(\beta)$$

Their solution has the form

$$v_{nj}^{\pm}(\alpha)=\mp\theta(\pm x_0)\frac{(-i\alpha)^{1-n}}{(-i\beta)^{2-n}}\delta_{3-n,j}\exp[i(\alpha x_0+\beta y_0)]\mp\left(\frac{\alpha}{\beta}\right)^{3-2n}v_{3j}^{\pm}(\alpha)\pm\frac{h_{4-n,j}^{-}(\beta)\pm h_{4-n,j}^{+}(\beta)}{2(-i\beta)^{2-n}(-i\alpha)^{n-1}}$$

Eliminating v_{nj} $(n=1,2)$ in the third relationship of (1.4), we arrive at the above-mentioned Riemann scalar problem in the function $v_{3j}(\alpha)$

$$p_{\alpha\beta}^{+}v_{3j}^{+}(\alpha)=p_{\alpha\beta}^{-}v_{3j}^{-}(\alpha)-(Q_{\alpha\beta}-G_{\alpha\beta})\quad(-\infty<\alpha<\infty) \qquad (1.5)$$
$$p_{\alpha\beta}^{\pm}=a_{22}^{\pm}g^{\pm}(\alpha,\beta)\bar{g}^{\pm}(\alpha,\beta),\quad g^{\pm}(\alpha,\beta)=(\alpha-\beta z_1^{\pm})(\alpha-\beta z_2^{\pm})$$
$$Q_{\alpha\beta}=\exp[i(\alpha x_0+\beta y_0)]\sum_{m=1}^{2}c_{m+1}^{\circ}(\alpha,\beta)\delta_{mj}$$
$$c_m^{\circ}(\alpha,\beta)=i\frac{\alpha^{m-1}}{\beta^{m-1}}[\theta(x_0)l_m^{+}(\alpha,\beta)+\theta(-x_0)l_m^{-}(\alpha,\beta)]$$

$$G_{\alpha\beta} = \sum_{m=2}^{5} c_m^-(\alpha, \beta)\, h_{mj}^-(\beta) + \sum_{m=2}^{3} c_m^+(\alpha, \beta)\, h_{mj}^+(\beta)$$

$$c_2^\pm = i\beta^{1/2}(l_1^+ \mp l_1^-), \quad c_3^\pm = -i\alpha^{1/2}(l_2^+ \mp l_2^-),$$

$$c_4^- = i\alpha^2\beta, \quad c_5^- = -i\alpha\beta^2$$

The known property $\operatorname{Im} z_k^\pm > 0$ /4/ of the roots of the characteristic polynomials $p_{\alpha\beta}^\pm$ involves the following fact.

Lemma. The function $q^-(z, \beta)/q^+(z, \beta)$ $(\bar q^-(z, \beta)/\bar q^+(z, \beta))$ is analytic for $\beta > 0$ in the lower half-plane $\operatorname{Im} z < 0$ (the upper half-plane $\operatorname{Im} z > 0$) and for $\beta < 0$ in the upper (lower) half-plane of the complex plane $z = \alpha + i\xi$, and its index on the line $z = \alpha$ is zero.

Relying on this assertion, we solve problem (1.5) and thereby problem (1.4) as well. The transforms h_{nj}^+ $(n = 2, 3)$, h_{nj}^- $(n = 2, 3, 4, 5)$ are contained in the expressions obtained for the transform of the fundamental matrix components. Since four functions from (1.1) are considered to be known on the line $x = 0$, the solution found in α should be inverted and conditions (1.3) utilized to eliminate the unknowns /8/. For instance, if the jump $H_n^-(y)$ $(n = 2, 3, 4, 5)$ are given on L, then by eliminating h_{nj}^+ we obtain the fundamental solution

$$u_{pj} = w_{pj}, \quad (p = 1, 2, 3), \quad \partial_2^1 u_{pj} = w_{pj}, \quad \partial_1^1 u_{pj} = w_{p+2,\, j} \ (p = 4, 5) \tag{1.6}$$

$$w_{pj}(x, y, x_0, y_0) = \frac{1}{\pi} \operatorname{Im} \sum_{n=1}^{2} \Bigg\{ \theta(x) N_{jn}^+ \Bigg[\sum_{m=2}^{5} c_{mn}^+ \int_L \frac{H_m^-(t)}{\mu_n^+ - t}\, dt +$$

$$\frac{\theta(x_0) R_{jn}^+}{\mu_n^+ - \mu_{n0}^+} + \sum_{k=1}^{2} \Phi_{jnk}^+ \Bigg] + \theta(-x) N_{jn}^- \Bigg[\sum_{m=2}^{5} C_{mn}^- \int_L \frac{H_m^-(t)}{\mu_n^- - t}\, dt +$$

$$\frac{\theta(-x_0) R_{jn}^-}{\mu_n^- - \mu_{n0}^-} - \sum_{k=1}^{2} \Phi_{jnk}^- \Bigg] \Bigg\}, \quad \Phi_{jnk}^\pm = \pm \frac{\theta(\pm x_0) K_{jnk}^\pm}{\mu_n^\pm - \bar\mu_{n0}^\pm} \mp$$

$$\frac{\theta(\mp x_0) L_{jnk}^\pm}{\mu_n^\pm - \mu_{n0}^\mp}$$

$$\mu_n^\pm = z_n^\pm x + y, \quad \mu_{n0}^\pm = z_n^\pm x_0 + y_0, \quad R_{jn}^\pm = l_j^\pm(z_n^\pm)\, k_n^\pm$$

$$K_{jnk}^\pm = \bar l_j^\pm(\bar z_k^\pm)\, b_{nk}^{\pm\pm}, \quad L_{jnk}^\pm = l_j^\mp(z_k^\mp)\, b_{kn}^{\pm\pm}, \quad N_{1n}^\pm = (z_n^\pm)^2$$

$$N_{2n}^\pm = 1, \quad N_{3n}^\pm = -z_n^\pm, \quad N_{4n}^\pm = l_2^\pm(z_n^\pm), \quad N_{5n}^\pm = l_1^\pm(z_n^\pm)$$

$$N_{6n}^\pm = z_n^\pm l_2^\pm(z_n^\pm), \quad N_{7n}^\pm = z_n^\pm l_1^\pm(z_n^\pm), \quad k_n^\pm = (-1)^n \times$$
$$[a_{22}^\pm \bar q^\pm(z_n^\pm, 1)(z_1^\pm - z_2^\pm)]^{-1}, \quad l_1^\pm(z) = (A_4^\pm + iA_1^\pm)z -$$
$$iA_3^\pm, \quad l_2^\pm(z) = A_4^\pm - iA_1^\pm + iA_2^\pm z$$

$$A_1^\pm = a_{22}^\pm \operatorname{Im}(z_1^\pm z_2^\pm), \quad A_2^\pm = a_{22}^\pm \operatorname{Im}(z_1^\pm + z_2^\pm),$$

$$A_3^\pm = a_{22}^\pm \operatorname{Im}(z_1^\pm z_2^\pm(\bar z_1^\pm + \bar z_2^\pm))$$

$$A_4^\pm = a_{12}^\pm - a_{22}^\pm \operatorname{Re}(z_1^\pm z_2^\pm), \quad b_{nk}^{++} = \frac{(-1)^{k+n}}{|z_1^+ - z_2^+|^2 d} \times$$
$$\left\{ \frac{d}{a_{22}^\pm(z_n^+ - \bar z_k^+)} + \bar l_1^+(\bar z_{k+1}^+) - l_1^-(\bar z_{k+1}^+) - z_{n+1}^+[\bar l_2^+(\bar z_{k+1}^+) - l_2^-(\bar z_{k+1}^+)] \right\}$$

$$b_{nk}^{+-} = \frac{(-1)^{k+n+1} d^{-1}}{(z_1^+ - z_2^+)(z_1^- - z_2^-)} \{\bar l_1^+(z_{k+1}^-) - l_1^-(z_{k+1}^-) - z_{n+1}^+[\bar l_2^+(z_{k+1}^-) -$$
$$l_2^-(z_{k+1}^-)]\}, \quad d = r_2^+ r_3^+ - (r_1^+)^2 - (r_4^-)^2 \ (r_n^\pm = A_n^+ \pm A_n^-)$$
$$(n = 1, 2, 3, 4)$$

The quantities $b_{nk}^{-\pm}$ are obtained from $b_{nk}^{+\pm}$ by replacing the indices $\pm$ in all the quantities by their opposites; we will not present the expressions for $C_{nm}^\pm$ which are analogous in form since they will not be used henceforth.

Analogous expressions are obtained for the components of the matrix U if other combinations of the functions from (1.1) are considered known.

The fundamental solution obtained enables us to give a mathematical formulation of different problems for a composite anisotropic plane containing defects both within and on the line separating the half-planes in the form of singular integral equations (or their stem). In particular, if there are no defects on the line of separation, we should set $H_m^-(y) \equiv 0$ $(m = 2, 3, 4, 5)$ in (1.6).

2. Reduction of the problem about a stringer to an integral equation. Let a stringer (a thin inclusion resistive to just tension of compression) be contacting along the whole length $S = (0, a)$ $(0 < a < \infty)$ with the right half-plane $(x \geqslant 0)$, loaded by a concentrated force P at the point $x = y = 0$ and located at an angle $\pi/2 - \varphi(|\varphi| \leqslant \pi/2)$ to the line of separation of the materials (Fig.1) A concetrated force $P^* = (P_1^*, P_2^*)$ is applied to the plane at a certain point (x_*, y_*). The shear stress $p(t)$ $(t \in S)$ orginating in the contact zone and satisfying the equilibrium condition

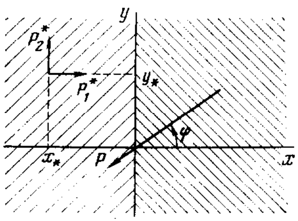

Fig.1

$$\int_S p(t)\,dt = -P \tag{2.1}$$

is considered unknown.

By using (1.6) and realizing the condition of compatibility of the deformation of the stringer and the plane, we arrive at a singular integral equation with a fixed singularity at the point $t = \tau = 0$ with respect to the contact stress desired

$$K_1[\chi](t) - c_0 R_1[\chi](t) = f(t) \quad (t \in I = [0, 1]) \tag{2.2}$$

$$\chi(t) = p(at), \quad c_0 = ac_*, \quad c_* = (ES_0 g)^{-1}, \quad f(t) = -f_1(at)$$

$$K_a[\chi](t) \equiv \frac{1}{\pi}\int_0^a \frac{\chi(\tau)}{t-\tau}\,d\tau + \operatorname{Im}\sum_{k,n=1}^{2} g_{kn}\frac{1}{\pi}\int_0^a \frac{\chi(\tau)\,d\tau}{t - e_{kn}\tau}$$

$$f_1(t) = \frac{q^{-1}}{\pi}\operatorname{Im}\sum_{k=1}^{2}\left\{\frac{\theta(x_0)\,g_k}{\tau - \xi_k} + \sum_{n=1}^{2}\left[\frac{\theta(x_*)\,g_{kn}^+}{t - \xi_{kn}^+} - \frac{\theta(-x_*)\,g_{kn}^-}{t - \xi_{kn}^-}\right]\right\}$$

$$R_a[\chi](t) \equiv \int_t^a \chi(\tau)\,d\tau, \quad q = -\frac{d\,(A_3^+\cos^2\varphi + A_1^+\sin 2\varphi + A_2^+\sin^2\varphi}{A_5^+ + (A_4^+)^2}$$

$$e_{kn} = \bar{\nu}_n \nu_k^{-1}, \quad \nu_k = z_k^+\cos\varphi + \sin\varphi, \quad \xi_k = \mu_{k*}^+ \nu_k^{-1}$$

$$\xi_{kn}^+ = \bar{\mu}_{n*}^+ \nu_k^{-1}, \quad \xi_{kn}^- = \mu_{n*}^- \nu_k^{-1}, \quad \mu_{n*}^{\pm} = z_n^{\pm} x_* + y_*$$

$$g_k = \lambda_k \varkappa_k^+ k_k, \quad g_{kn}^+ = \lambda_k \bar{\varkappa}_n^+ b_{kn}^{++}, \quad g_{kn}^- = \lambda_k \varkappa_n^- b_{kn}^{+-}$$

$$q_{kn} = b_{kn}^{++}\lambda_k \bar{\lambda}_n q^{-1}, \quad \lambda_k = l_1^+(z_k^+)\cos\varphi + l_2^+(z_k^+)\sin\varphi$$

$$\varkappa_k^{\pm} = P_1^* l_{1k}^{\pm} + P_2^* l_{2k}^{\pm}, \quad l_{nk}^{\pm} = l_n^{\pm}(z_k^{\pm})$$

(E and S_0 are Young's modulus and the cross-sectional area of the stringer).

We will investigate (2.2) in the Banach space of the functions $L_p(I, \rho)$ $(1 < p, \ \rho(t) = t^\delta (1-t)^\gamma, \ p - 1 > \delta > -1 + p\operatorname{Re}\nu, \ p - 1 > \gamma > -1 + p/2, 0 \leqslant \operatorname{Re}\nu < 1)$ with norm introduced in /8/, for example.

3. Analysis of the behaviour of the solution at the endpoints.

It can be shown that the solution in the neighbourhood of one possesses the asymptotic form

$$\chi(t) = O((1-t)^{-1/2}), \quad t \to 1 - 0 \tag{3.1}$$

Taking account of the property of integrability of the solution that results from condition (2.1), we assume that the asymptotic representation is valid ($M_n(t)$ is a certain polynomial of exact degree n)

$$\chi(t) = O(t^{-\nu} M_n(\ln t)), \ t \to 0, \ 0 \leqslant \operatorname{Re}\nu < 1, \ n = 0, 1, 2, \ldots \tag{3.2}$$

To determine ν and n we set $\chi(t) = \chi_*(t)\,t^{-\nu} M_n(\ln t)$ $(\chi_*(0) \neq 0, \ |\chi_*(0)| < \infty)$. Let $\nu \neq 0$, then by using the relationships (8.35) and (8.36) from /9/, an asymptotic formula can be obtained for the operator K_1

$$K_1[\chi](t) = t^{-\nu}\chi_*(0)\sum_{k=0}^{n}\frac{(-1)^{k+1}}{k!}G_*^{(k)}(\nu)\,M_n^{(k)}(\ln t) + \Omega_1(t) \tag{3.3}$$

$$\sin\nu\pi\, G_*(\nu) = \cos\nu\pi + \frac{i}{2}\sum_{n,k=1}^{2}\left[(-e_{kn})^{\nu-1} g_{kn} - (-\bar{e}_{kn})^{\nu-1}\bar{g}_{kn}\right]$$

$$M_n^{(k)}(x) = \frac{d^k}{dx^k}M_n(x)$$

Using (3.3) and also taking account of the boundedness of the functions $f(t)$ and $R_1[\chi](t)$ for $\chi(t) \in L_p(I, \rho)$ on I, we obtain the following relationship from (2.2)

$$\Omega_2(t) = t^{-\nu}\chi_*(0)\sum_{k=0}^{n} G_*^{(k)}(\nu)\binom{k}{n} M_n^{(k)}(\ln t)(-1)^{k+1}\frac{1}{k!} \tag{3.4}$$

Here $\Omega_k(t)$ $(k = 1, 2)$ are certain functions possessing the property

$$\Omega_k(t) = O(t^{-\nu+\varepsilon}), \quad t \to 0, \quad \varepsilon > 0 \tag{3.5}$$

It follows directly from relationships (3.4) and (3.5)

$$G_*^{(k)}(\nu) = 0, \ (k = 0, 1, \ldots, n) \tag{3.6}$$

If $\nu = 0$ in (3.2), then by using (8.30) and (8.31) from /9/, we obtain an asymptotic formula in place of (3.3) (B_j are Bernoulli numbers, $\Omega_*(t)$ is bounded at zero)

$$K_1[\chi](t) = \frac{\chi_*(0)}{\pi} \sum_{k=0}^{k} M_n^{(k-1)}(\ln t) \frac{(-1)^{k+1}}{k!} \sum_{j=0}^{k} \lambda_{j,k} G_{s0}^{(k-j)} + \Omega_*(t)$$

$$\lambda_{j,k} = \binom{k}{j} (2^j - 2)\pi^j B_j$$

$$G_{s0}^{(k)} = \frac{d^k}{d\eta^k} [\sin \pi\eta G_*(\eta)]|_{\eta=0}$$

which, when utilized in the same way as (3.6), results in the relationship

$$G_{s0}^{(k)} = 0 \quad (k = 0, 1, \ldots, n)$$

Comparing the last relationship with (3.6), we can find that $-\nu$ in (3.2) is the $(n+1)$-tuple root from the strip $0 \leqslant \operatorname{Re}\eta < 1$ (including the point $\nu = 0$ also) for the transcendental equation

$$G_0(\eta) \equiv \sin \pi\eta G_*(\eta) = 0 \tag{3.7}$$

It is not possible to show the existence and uniqueness of such a root in the general case. However, on satisfying the condition

$$0 < d_* < 1 \tag{3.8}$$

$$d_* = \frac{A_5^+ + (A_4^+)^2}{d} \left(\frac{A_5^-}{A_5^+} + \frac{k^-}{k^+} \right), \quad k^\pm = A_3^\pm \cos^2\varphi + A_1^\pm \sin 2\varphi + A_2^\pm \sin^2\varphi$$

the presence and uniqueness of a root of the Eq.(3.7) in the strip $0 \leqslant \operatorname{Re}\nu < 1$ can be ensured. Condition (3.8) is satisfied for known anisotropic materials (including orthotropic and isotropic), where the root mentioned turns out to be real. Therefore, the index ν of the singularity and the degree of the logarithmic polynomial in the asymptotic form (3.2) of the solution of (2.2) is determined by using the asymptotic properties of Cauchy-type integrals, despite the assertion made in /3/.

The incorrectness of this assertion, as well as of the results in /2/, is related to the fact that a relationship of the type (3.4) was multiplied, without any justification, by a function whose order of zero agrees as $t \to 0$ with the order of the singularity of the highest term in the principal part of the asymptotic form of the desired function. Hence, as $t \to 0$ terms having a stronger singularity than is controlled by condition (3.5) were lost. We note that this same procedure was used in /10/. However, it did not result there in an erroneous result since it was assumed that the behaviour of the solution is only of a power nature.

4. The exact solution of (2.2) for $c_0 = 0$ and its solvability in the general case. We consider the integral equation

$$K_1[\chi](t) = f(t) \quad (t \in I) \tag{4.1}$$

which is obtained from (2.2) if we set $c_0 = 0$ and which corresponds to the problem of a finite inextensible stringer. To obtain the exact solution we predetermine (4.1) on the whole semi-axis.

$$K_\infty[\chi_-](t) = f_-(t) + \chi_+(t) \quad (0 < t < \infty) \tag{4.2}$$

$$\chi_-(t), f_-(t) = \begin{cases} \chi(t), f(t) & (0 < t < 1) \\ 0 & (1 < t) \end{cases}$$

($\chi_+(t)$ is a certain function that equals zero in the interval $(0, 1)$).

Analysis of the behaviour of $\chi_\pm(t)$ taking (3.1) and (3.2) into account shows

$$\chi_-(t) = O(t^{-\nu} M_n(\ln t)), \quad t \to 0; \quad \chi_-(t) = O((1-t)^{-1/2}), \quad t \to 1-0$$

$$\chi_+(t) = O((t-1)^{-1/2}), \quad t \to 1+0; \quad \chi_+(t) = O(t^{-1}), \quad t \to \infty$$

This implies the analyticity of the Mellin transform

$$X_+(s) = \int_1^\infty \chi_+(t) t^{s-1} dt, \quad X_-(s) = \int_0^1 \chi(t) t^{s-1} dt$$

for $\operatorname{Re} s < 1$ and $\operatorname{Re} s > \operatorname{Re}\nu$, respectively.

Applying the Mellin integral transformation to (4.2), using the convolution theorem, and taking account of the integrals (2.2.4.25) and (2.2.4.26) in /11/, we arrive at the Riemann problem

$$X_+(\eta) = G(\eta) X_-(\eta) + F_-(\eta) \quad (\eta \in \Gamma: \operatorname{Re} s = \gamma_1) \tag{4.3}$$

$$\operatorname{Re} \nu < \gamma_1 < 1, \quad G(\eta) = G_*(\eta), \quad F_-(\eta) = -\int_0^1 f(t) t^{\eta-1} dt$$

where $X_\pm(\eta)$ are analytically continuable functions in the half-planes Ω_+ and Ω_-, respectively (Ω_+: $\operatorname{Re} s < \gamma_1$; Ω_-: $\operatorname{Re} s > \gamma_1$). Relationship (2.1) and a theorem of Abel type for problem (4.3) supply the additional conditions

$$X_-(1) = -P/a, \quad X_\pm(s) = O(s^{-1/2}) \quad (s \to \infty, s \in \Gamma) \tag{4.4}$$

As a result of some reduction, the coefficient of problem (4.3) allows of the representation

$$G(\eta) = K(\eta) G_1(\eta), \quad G_1(\eta) = \tag{4.5}$$

$$\frac{\cos \pi\eta - g_1(\eta - 1) - g_2(\eta - 1) - g_2(1 - \eta)}{\cos \pi\eta - 1}$$

$$K(\eta) \equiv \operatorname{tg} \frac{\pi\eta}{2} = \frac{K_-(\eta)}{K_+(\eta)}, \quad K_+(\eta) = \frac{\Gamma(1 + \eta/2)}{\Gamma(1/2 + \eta/2)}, \quad K_-(\eta) = \frac{\Gamma(1/2 + \eta)}{\Gamma(\eta/2)}$$

The functions $K_\pm(s)$ are analytic in the domains Ω_+ and Ω_-, respectively, and the following asymptotic form holds:

$$K_\pm(s) = O((\mp s/2)^{1/2}), \quad s \to \infty, \quad s \in \Omega_\pm \tag{4.6}$$

The function $G_1(\eta)$ possesses the following properties (H_Γ is the class of functions satisfying the Hölder condition on Γ):

$$G_1(s) \neq 0 \quad (s \in \Omega = \Omega_- \cap \Omega_+) \tag{4.7}$$

$$\operatorname{Ind} G_1(\eta) = 0, \ \ln G_1(\eta) \in H_\Gamma \quad (\eta \in \Gamma)$$

Therefore, well-known formulas /9/ can be used to factorize $G_1(\eta)$. Furthermore, taking account of properties (4.4), (4.6) and applying Liouville's theorem, we obtain a solution of problem (4.3), and therefore of (4.1) also

$$\chi(t) = \frac{1}{2\pi i} \int_\Gamma \frac{\Phi^+(\eta)}{K_+(\eta)} \frac{A - \Psi^-(\eta)}{G(\eta)} t^{-\eta} d\eta \quad (t \in I) \tag{4.8}$$

$$\Phi(s) = \exp\left[\frac{1}{2\pi i} \int_\Gamma \frac{\ln G_1(t)}{t - s} dt\right], \quad \Psi(s) =$$

$$\frac{1}{2\pi i} \int_\Gamma \frac{K_+(t)}{\Phi^+(t)} F_-(t) \frac{dt}{t - s}$$

$$A = -\frac{P\sqrt{d_*}}{a\pi} + \Psi^-(1)$$

Taking account of the boundedness of the function $f(t)$ $(t \in I)$, it can be shown that the highest term in the asymptotic form $\chi(t)$ is determined as $t \to 0$ by the residue of the integrand at the point (ν), the first $(n + 1)$-tuple pole of the function $G_0(\eta)$ to the left of the contour Γ, and has the form (3.2).

It can be seen that all the above remains true if the arbitrary function $f(t) \in L_p(I, \rho)$ possessing the properties

$$f(t) = O((1 - t)^{\varepsilon - 1/2}), \quad t \to 1 - 0, \quad f(t) = O(t^{-\nu_0}), \quad t \to 0 \tag{4.9}$$

$$\varepsilon > 0, \quad \operatorname{Re} \nu_0 < \operatorname{Re} \nu$$

is taken as the right-hand side of (4.1).

According to (4.8), the homogeneous Eq.(4.1) $(f(t) = 0)$ (corresponding to the case $P_* = 0$) has one linearly dependent solution. It is shown analogously that the conjugate equation $K_1^*[\chi^*](t) = 0$ (K_1^* is the operator conjugate to K_1) in the space $L_q(I, \rho_*)$ $(q = p/(p - 1)$, $\rho_* = t^\sigma (1 - t)^\omega$, $\sigma = -\delta q/p$, $\omega = -\gamma q/p)$ conjugate to the space $L_p(I, \rho)$ has only a zero solution under the condition (3.8).

The following is therefore proved.

Theorem 1. Under condition (3.8), Eq.(4.1) is a Noether equation in $L_p(I, \rho)$ $(1 < p < \infty$, $\rho = t^\delta (1 - t)^\gamma$, $p - 1 > \delta > -1 + p \operatorname{Re} \nu$, $p - 1 > \gamma > p/2 - 1)$ and its index equals one. Under the additional condition (2.1), Eq.(4.1) has the unique solution (3.8) that possesses the asymptotic properties (3.1), (3.2) under the conditions (4.8).

The continuity and boundedness of the operator R_1 in $L_p(I, \rho)$ as well as Theorem 3.4 from /12/ and 4.1 enable us to formulate the following assertion.

Theorem 2. Under condition (3.8), Eq.(2.2) is a Noether equation in $L_p(I, \rho)$, its index equals one, and under the additional condition (2.1) it has a unique solution that possesses the asymptotic properties (3.1), (3.2) under the conditions (4.9).

We note that questions of the solvability of singular integral equations with fixed singularities were apparently first investigated in /8/. In particular, the solvability is shown for the integral equation of the problem examined in /10/.

5. Exact solution for a semi-infinite stringer. Here (2.2) takes the form ($a = \infty$)

$$K_\infty [p](t) - c_* R_\infty [p](t) = -f_1(t) \quad (0 < t < \infty) \tag{5.1}$$

As in the case of the finite stringer, the operator K_∞ evidently yields exhaustive information about the nature of the behaviour of the solution as $t \to 0$, and therefore, the following asymptotic representation holds (see (3.2))

$$p(t) = O(t^{-\nu} M_n(\ln t)), \quad t \to 0 \tag{5.2}$$

As $t \to \infty$ the operator R_∞ exerts a decisive influence on the nature of the behaviour of the solution. Thus if the stringer is inextensible ($c_* = 0$), then the solution of (5.1) that vanishes at infinity as $t^{2-\nu}$ has the form (it is here assumed that $P = 0$)

$$p(t) = \frac{1}{2\pi i} \int\limits_\gamma \frac{F(\eta)}{G(\eta)} t^{-\eta} d\eta$$

$$(T : \operatorname{Re} \eta = \gamma_0, \operatorname{Re} \nu < \gamma_0 < 2 - \operatorname{Re} \nu)$$

$$F(\eta) = -\int\limits_0^\infty f_1(t) t^{\eta-1} dt$$

(We note that the homogeneous Eq.(5.1) has only a zero solution for $c_* = 0$). If $c_* \neq 0$, then for (5.1) to be satisfied as $t \to \infty$ it is necessary to impose the following condition on the solution: $p(t) = O(t^{-1-\tau}), \tau > 1, t \to \infty$. Taking account of (5.2) and Theorem 3.1 of /13/, that latter enables us to obtain the first-order difference equation

$$X(\eta + 1) = \frac{\eta}{c_*} G(\eta) X(\eta) - \frac{\eta}{c_*} F(\eta), \quad \eta \in \Gamma = \{\operatorname{Re} s = \gamma_1\} \tag{5.3}$$

for the Mellin transform of the desired function that is analytic in the strip $W = W_+ \cup W_-$ ($W_+ = \{s \mid \lambda \leqslant \operatorname{Re} s < \gamma_2\}$, $W_- = \{s \mid \gamma_2 < \operatorname{Re} s < \lambda + 1\}$, $\lambda \in (\operatorname{Re} \nu, 1)$, $\gamma_2 \in (\lambda, 2 - \operatorname{Re} \nu)$) and tends to zero as $|\operatorname{Im} s| \to \infty$ within this strip.

The represenation (4.5) holds for $G(s)$. Here (see /14/, say)

$$\frac{s}{c_*} \operatorname{tg} \frac{\pi s}{2} = \frac{K_*(s)}{K_*(s+1)}, \quad K_*(s) = \frac{\sin \frac{\pi s}{2}}{\Gamma(s)} c_*^s \quad (s \in W)$$

The function $K_*(s)$ is analytic in W, and the properties (4.7) hold for the function $G_1(s)$ that is analytic in the strip W_+. Therefore, taking account of the results in /15, 16/, we write the canonical solution of the homogeneous problem (5.3) in the form

$$X_0(s) = Y(s)[G_1(s) K_*(s)]^{-1}, \quad s \in W_+; \quad X_0(s) = Y(s) K_*^{-1}(s)$$

$$s \in W_-$$

$$Y(s) = \exp\left\{\frac{1}{2i} \int\limits_\Gamma \ln G_1(t) \operatorname{ctg} \pi (t - s) \, dt\right\}$$

We note that another approach was applied to the solution of an equation of the form (4.4) in /3/.

Following /16/, we represent the solution of the inhomogeneous problem (5.3) as follows (A is a constant determined from the additional condition (2.1)):

$$X(s) = X_0(s)[A + \cos \pi s \, Z_1(s)] \tag{5.4}$$

$$Z_1(s) = Z(s) - F(s), \ s \in W_+; \ Z_1(s) = Z(s), \ s \in W_-$$

$$Z(s) = \frac{1}{2i} \int\limits_\Gamma F_*(t) \sin^{-1}(t - s) \, dt, \quad F(s) = \frac{s}{c_*} \frac{F(s)}{X_0(s+1)} \in H_\Gamma$$

Passing to the limit as $s \to \Gamma$ in (5.4) and using the Sokhotskii formula, we obtain (5.1) in two equivalent forms after inversion

$$p(t) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{\Gamma(\eta)}{\sin \frac{\pi \eta}{2}} \frac{Y^\pm(\eta)}{G_1^{(0,5 \pm 0,5)}(\eta)} \times [A + \cos \pi \eta Z^-(\eta)] \left(\frac{t}{c_*}\right)^{-\eta} d\eta \tag{5.5}$$

$$A = -P c_* \sqrt{d_*} + Z^-(1)$$

Taking account of the analyticity properties of the functions $Y^\pm(s)$ in the strips $W_\pm$, respectively, the asymptotic representation (5.2) is easily confirmed, and the order of decrease of the solution at infinity, $\tau = -1$, is easily refined.

6. Numerical analysis of the transcendental equation. The roots of the transcendental Eq. (3.7), that depend on the elastic constants of the half-planes and the angle of rotation of the stringer, play an important part in the solutions of the problems examined above. A numerical analysis confirmed that (3.7) has one real prime root in the interval $(0, 1)$ for different combinations of the quantities to be varied when (3.8) is satisfied.

In particular, results of calculations of the roots of (3.7) in the interval $(0, 1)$ are presented in Figs.2-4 for a plane from a boron epoxy composite (the left half-plane) with the parameters $E_1^- = 4\cdot10^3$ MN/m^2, $E_2^- = 4\cdot10^4$ MN/m^2, $G_1^- = 1.5\cdot10^4$ MN/m^2 $\nu_1^- = 0.25$, $z_1^- = 5.08i$, and $z_2^- = 0.62i$ and for a graphite epoxy composite (the right half-plane) with the parameters $E_1^+ = 4\cdot10^3$ MN/m^3, $E_2^- = 1.6\cdot10^4$ MN/m^2, $G_1^+ = 0.8\cdot10^4$ MN/m^2, $\nu_1^+ = 0.25$, $z_1^+ = 6.99i$, $z_2^+ = 0.71i$. The computations were performed for different values of the angles formed by the principal axes of the material elasticity of the right (ψ_r) and left (ψ_l) half-planes with the axis x and the values of the stringer slope $0 \leqslant \varphi \leqslant \pi/2$. Curves 1-6 correspond to the values $\psi_r = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$.

It is seen that a minimum value of the index of the singularity, in absolute value, can be achieved by a suitable selection of the angles ψ_l, ψ_r, φ. An analogous situation occurs if certain of the parameters mentioned are fixed for any reasons whatever.

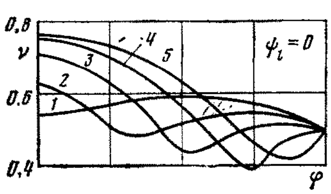

Fig.2

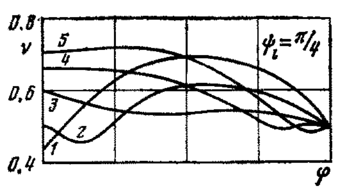

Fig.3

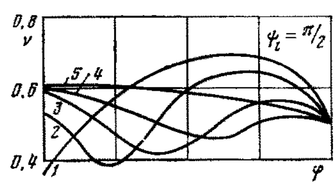

Fig.4

The following regularity was also detected during the calcuations: if the left half-plane is stiffer than the right in its elastic characteristics, then the index of the singularity will always be greater than 0.5 and $\nu \to 1$ as $E_k^- \to \infty$ $(k = 1, 2)$; if the left half-plane is weaker, then $\nu < 0.5$. In particular, if $E_k^- \to 0$, the quantity ν tends to the appropriate indices for the problem of a stringer emerging on the boundary of an anisotropic half-plane (see Fig.2 in /1/).

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SPHERICALLY LAYERED INCLUSIONS IN A HOMOGENEOUS ELASTIC MEDIUM*

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A three-dimensional homogeneous and isotropic elastic medium is considered that contains an isolated inhomogeneity (inclusion) in the shape of a sphere. It is assumed that the elastic moduli of the medium within the sphere depend only on the distance r to the centre of the inclusion. It is shown that in the case of a constant external field the problem of the equilibrium of a medium with an inhomogeneity reduces to a system of ordinary differential equations in three scalar functions of the variable r. An inhomogeneity with a piecewise-constant dependence of the elastic moduli on r (a spherically layered inclusion) is examined in detail. In this case, an effective calculational algorithm is proposed to construct the solution of the problem. The solution of the problem of one inclusion is then utilized to determine the effective elastic moduli of a medium with a random set of spherically layered inclusions and the estimates of the stress concentration at individual inhomogeneities. The method of an effective (selfconsistent) field is used to take account of interaction between the inclusions.

The problem of a spherically layered inclusion in a homogeneous elastic medium was solved /1-3/ for particular forms of the constant external field. The method proposed below enables us, within the framework of a single scheme, to examine both spherically layered inclusions with practically any number of layers and inclusions with elastic moduli varying continuously along the radius for an arbitrary homogeneous external stress (strain) field.

1. The integral equation of the problem. In an infinite homogeneous medium with the elastic modulus tensor c_0 let there be an isolated inhomogeneity occupying a finite domain V whose characteristic function is $V(x)$, where $x(x_1, x_2, x_3)$ is a point of the medium. We shall consider the elastic modulus tensor $c(x)$ to be a piecewise-smooth function of the coordinates with the domain V. We examine the deformation of the medium $\varepsilon(x)$ under the effect of self-equilibrated forces at infinity and certain mass forces.

Let $\varepsilon_0(x)$ denote the external field of deformations that would exist in a medium when there are no inhomogeneities and the same loading conditions. It is known /4/ that a perturbation of the strain tensor $\varepsilon_1(x) = \varepsilon(x) - \varepsilon_0(x)$ in a medium with an inhomogeneity will satisfy the equation

$$\varepsilon_{1\alpha\beta}(x) + \int_V K_{\alpha\beta\lambda\mu}(x-x') c_1^{\lambda\mu\nu\rho}(x') \varepsilon_{1\nu\rho}(x')\,dx' = -\int_V K_{\alpha\beta\lambda\mu}(x-x') c_1^{\lambda\mu\nu\rho}(x') \varepsilon_{0\nu\rho}(x')\,dx', \quad c_1(x) \equiv c(x) - c_0 \tag{1.1}$$

The kernel $K(x)$ of the integral operator K in this equation is expressed in terms of the second derivatives of Green's function $G(x)$ for the medium c_0

$$K_{\alpha\beta\lambda\mu}(x) = -(\nabla_\alpha \nabla_\lambda G_{\beta\mu}(x))_{(\alpha\beta)(\lambda\mu)} \tag{1.2}$$

The function $G(x)$ satisfies the well-known equation (δ_β^α is the Kronecker delta, and $\delta(x)$ is the delta function)

$$\nabla_\alpha c_0^{\alpha\beta\lambda\mu} \nabla_\lambda G_{\mu\nu}(x) = -\delta_\nu^\beta \delta(x) \tag{1.3}$$

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